ISRAEL JOURNAL OF MATHEMATICS **133** (2003), 177–188

THE HOCHSCHILD HOMOLOGY OF FATPOINTS

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ABSTRACT

This paper calculates the Hochschild homology of fatpoints—rings of the form $k[x_1, x_2, \ldots, x_d]/\mathfrak{m}^a$ where k is a field of characteristic zero and $\mathfrak{m} = (x_1, x_2, \ldots, x_d)$. The calculation includes the multiplicative structure induced by the shuffle product. The answer is given in terms of the homology of tori relative to their a-diagonal. Since the rings in question are monoidal, this calculation also determines their topological Hochschild homology. By a theorem of Goodwillie, the dimensions of the cyclic homology groups of these rings are determined as well.

0. Introduction

The purpose of this paper is to calculate the Hochschild homology of rings of the form $k[x_1, \ldots, x_d]/\mathfrak{m}^a$, where k is a field of characteristic 0 and $\mathfrak{m}$ is the maximal ideal $(x_1, \ldots, x_d)$. The result, Corollary 3.2 below, is an algebra isomorphism

$$\mathrm{HH}_*(k[x_1, \ldots, x_d]/\mathfrak{m}^a) \cong \bigoplus_{w=0}^{\infty} \mathrm{H}_*(\mathbb{T}^w, \Delta_a; k) \odot_{k[\Sigma_w]} (k^d)^{\otimes w},$$

where Δ_a is the set of all points in the w-torus $\mathbb{T}^w$ which have a or more identical coordinates. The Hochschild homology complex of $k[x_1, \ldots, x_d]/\mathfrak{m}^a$ can readily be seen to split as a direct sum over all $w \geq 0$ of the subcomplexes consisting of sums of tensor monomials of total weight w. In the result above, the summand corresponding to w gives the homology of the weight w subcomplex of the Hochschild complex. The product on Hochschild homology is induced, as usual,

Received September 4, 2001 and in revised form February 5, 2002

by the shuffle product, and the product on the right hand side of the equation is induced by the map of pairs $(\mathbb{T}^w, \Delta_a) \times (\mathbb{T}^{w'}, \Delta_a) \to (\mathbb{T}^{w+w'}, \Delta_a)$ coming from the standard identification $\mathbb{T}^w \times \mathbb{T}^{w'} \cong \mathbb{T}^{w+w'}$, tensored with the identification $(k^d)^{\otimes w} \otimes (k^d)^{\otimes w'} \cong (k^d)^{\otimes w+w'}$.

Computational work on the Hochschild homology of commutative rings has mostly focused on rings which are not too far from being complete intersections. The basic computational technique was developed by Tate [8] and exploited by many others: see, for example, [10], [1], [3], and [5]. But the result here is quite different. The ring examined here is at the opposite extreme from being a complete intersection. The proof uses simplicial methods and a comparison with a particular simplicial decomposition of tori rather than Koszul complexes. Another exception to the Koszul-type calculations is the work by S. Geller, L. Reid and C. Weibel [2] describing the Hochschild homology of the coordinate axes. Perhaps this computation and the computation in this paper are both special cases of a general theory of the Hochschild homology of quotients of $k[x_1, \dots, x_d]$ by monomial ideals.

Rings of the form $k[x_1, \dots, x_d]/\mathfrak{m}^a$ are monoidal over k, in the sense that they have a basis $\mathfrak{B}$ over k such that the identity of the ring is in $\mathfrak{B}$ and $\mathfrak{B} \cup \{0\}$ is closed under multiplication. In such cases, by Lemma 6.1 in [4] the topological Hochschild homology spectrum splits as

$$\mathbf{THH}(k[\mathfrak{B}]) \simeq \mathbf{THH}(k) \wedge |N^{\mathrm{cy}}_{\cdot}\mathfrak{B}|_+$$

where $|N^{\mathrm{cy}}_{\cdot}\mathfrak{B}|$ is a space whose homology with coefficients in any ring A is $\mathrm{HH}_*(A[\mathfrak{B}])$. Since the topological Hochschild homology spectrum of any ring is a product of Eilenberg Mac Lane spectra and k is a field, we get that the stable homotopy groups of the topological Hochschild homology spectrum THH_* satisfy

$$\mathrm{THH}_*(k[\mathfrak{B}]) \cong \mathrm{THH}_*(k) \otimes \mathrm{HH}_*(k[\mathfrak{B}]).$$

Thus this paper also calculates $\mathrm{THH}_*(k[x_1, \dots, x_d]/\mathfrak{m}^a)$ in terms of $\mathrm{THH}_*(k)$.

Also, since the ring $k[x_1, \dots, x_d]/\mathfrak{m}^a$ is graded, by a theorem of T. Goodwillie (see, for example, [9] 9.9.1), the long exact sequence relating Hochschild homology and cyclic homology in each positive weight w splits into short exact sequences

$$0 \to \mathrm{HC}_{p-1}(k[x_1, \dots, x_d]/\mathfrak{m}^a)_w \to \mathrm{HH}_p(k[x_1, \dots, x_d]/\mathfrak{m}^a)_w \\ \to \mathrm{HC}_p(k[x_1, \dots, x_d]/\mathfrak{m}^a)_w \to 0.$$

Since in weight zero we have Hochschild homology and cyclic homology only in dimension zero, both isomorphic to k, this gives us a short exact sequence

$$0 \to \mathrm{HC}_{p-1}(k[x_1,\dots,x_d]/\mathfrak{m}^a) \to \mathrm{HH}_p(k[x_1,\dots,x_d]/\mathfrak{m}^a) \to \mathrm{HC}_p(k[x_1,\dots,x_d]/\mathfrak{m}^a) \to 0$$

for every $p \geq 0$, and means that we can calculate the dimensions of the cyclic homology of $k[x_1,\dots,x_d]/\mathfrak{m}^a$ inductively from the calculation given here of its Hochschild homology.

ACKNOWLEDGEMENT: I would like to thank M. Larsen for several useful conversations, the referee for several useful suggestions, and L. Avramov for asking about multiplicative structure.

1. Notations

In this section we will introduce the notation required for the Hochschild homology calculation later in the paper. Given a commutative ring k and an associative, unital k-algebra A, we look at the Hochschild complex of A over k

$$\mathrm{CH}_n(A) = A^{\otimes n+1}$$

with ∂: $\mathrm{CH}_n(A) \to \mathrm{CH}_{n-1}(A)$ equal to $\sum_{i=0}^{n}(-1)^i d_i$ where

$$d_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n, \quad 0 \leq i < n,$$
$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_n a_0 \otimes a_1 \otimes a \cdots \otimes a_{n-1}.$$

All tensors are taken over k.

The homology of the complex $(\mathrm{CH}_*, \partial)$ is called the Hochschild homology of A over k, $\mathrm{HH}_*(A)$.

For $0 \leq i \leq n$, we can also look at s_i: $\mathrm{CH}_n(A) \to \mathrm{CH}_{n+1}(A)$ given by

$$s_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

Now $\mathrm{CH}_.(A)$ equipped with the d_i and the s_i is a simplicial abelian group, and the homology of the associate complex $(\mathrm{CH}_*, \partial)$ will not, by the theory of simplicial abelian groups (e.g., [6]), change if we quotient out by the degeneracies and look at

$$\widetilde{\mathrm{CH}}_n(A) = \mathrm{CH}_n(A) / \sum_{i=0}^{n-1} \mathrm{im}(s_i\colon \mathrm{CH}_{n-1}(A) \to \mathrm{CH}_n(A)).$$

$\widetilde{\mathrm{CH}}_n(A)$ is the quotient of $\mathrm{CH}_n(A)$ by the sums of all tensor monomials which have 1 in one of the positions other than the zeroth position.

Consider now the case $A = k[x_1, \dots x_d]$. Then there is an obvious basis for each $\mathrm{CH}_n(A)$ consisting of tensor monomials each of whose positions contains a (possibly empty) product of various x_i. This suggests a decomposition

$$\mathrm{CH}_n(k[x_1,\dots,x_d]) \cong \bigoplus_{(w_1,\dots,w_d)\in\mathbb{N}^d} \mathrm{CH}_n^{x_1^{w_1}x_2^{w_2}\dots x_d^{w_d}}(k[x_1,\dots,x_d])$$

where each $\mathrm{CH}_n^{x_1^{w_1}x_2^{w_2}\dots x_d^{w_d}}(k[x_1,\dots,x_d])$ is the linear span of those basis elements which, if we total up the occurrences of the variables in all the positions, have w_i occurrences of x_i for all i.

Observe that the boundary maps d_i and the face maps s_i respect this decomposition, and so

$$\mathrm{HH}_n(k[x_1,\dots,x_d]) \cong \bigoplus_{(w_1,\dots,w_d)\in\mathbb{N}^d} \mathrm{HH}_n^{x_1^{w_1}x_2^{w_2}\dots x_d^{w_d}}(k[x_1,\dots,x_d]).$$

Observe also that if we quotient the complex out by some basis elements, this decomposition still makes sense (we only need to make sure that the d_i will be well-defined). Thus we can apply this decomposition to the reduced Hochschild complex $\widetilde{\mathrm{CH}}_*(k[x_1,\dots,x_d])$, or to the reduced or unreduced Hochschild complex of a quotient ring $k[x_1,\dots,x_d]/I$ where I is the ideal generated by some set of monomials in the x_i.

For any algebra $k[z_1,\dots,z_b]$ on symbols $z_1,\dots,z_b$, we let $\mathfrak{m}$ denote the maximal ideal $(z_1,\dots,z_b)$.

2. Reduction to the calculation of $\mathrm{HH}_*^{x_1\cdots x_d}(k[x_1,\dots,x_d]/\mathfrak{m}^a)$

The goal of this paper is to calculate

$$\mathrm{HH}_n(k[x_1,\dots,x_d]/\mathfrak{m}^a) \cong \bigoplus_{(w_1,\dots,w_d)\in\mathbb{N}^d} \mathrm{HH}_n^{x_1^{w_1}x_2^{w_2}\dots x_d^{w_d}}(k[x_1,\dots,x_d]/\mathfrak{m}^a)$$

when k is a field of characteristic 0. In this section we will show

THEOREM 2.1: *If k is a field of characteristic 0, then for every $n \geq 0$*

$$\mathrm{HH}_n(k[x_1,\dots,x_d]/\mathfrak{m}^a) \cong \bigoplus_{w=0}^{\infty} \mathrm{HH}_n^{y_1\cdots y_w}(k[y_1,\dots,y_w]/\mathfrak{m}^a) \otimes_{k[\Sigma_w]} (k^d)^{\otimes w} \tag{2.1.1}$$

where Σ_w acts on $k[y_1,\dots,y_w]/\mathfrak{m}^a$ on the right by permuting the y_i and on $(k^d)^{\otimes w}$ on the left by permuting tensor factors.

Thus it suffices to look at the case where each variable occurs exactly once, but we need to know the answer in that case for any number of variables. Recall from the end of Section 1 that on the left hand side of equation (2.1.1), $\mathfrak{m} = (x_1, \ldots, x_d)$, while on its right side $\mathfrak{m} = (y_1, \ldots, y_w)$.

Definition 2.2: For $(w_1, \ldots, w_d) \in \mathbb{N}^d$, we let $G_{w_1,\ldots,w_d}$ be the stabilizer of the vector $(\underbrace{1,\ldots,1}_{w_1}, \underbrace{2,\ldots,2}_{w_2}, \ldots, \underbrace{d,\ldots,d}_{w_d})$ under the standard left Σ_w action (where $w = w_1 + \cdots + w_d$).

Throughout this paper, $\mathbb{N}$ denotes the set of non-negative integers. Note that $G_{w_1,\ldots,w_d} \cong \Sigma_{w_1} \times \Sigma_{w_2} \times \cdots \times \Sigma_{w_d}$.

LEMMA 2.3: *There is an isomorphism of complexes*

$$(2.3.1) \quad \begin{aligned} &\mathrm{CH}_*^{x_1^{w_1} x_2^{w_2} \cdots x_d^{w_d}} (k[x_1, \ldots, x_d]/\mathfrak{m}^a) \\ &\quad \cong \mathrm{CH}_*^{y_1 \cdots y_w} (k[y_1, \ldots, y_w]/\mathfrak{m}^a) \otimes_{k[\Sigma_w]} k[\Sigma_w / G_{w_1,\ldots,w_d}] \end{aligned}$$

where $w = w_1 + \cdots + w_d$, arising from the fact that they are both isomorphic to the coinvariants of the $G_{w_1,\ldots,w_d}$-action on $\mathrm{CH}_*^{y_1 \cdots y_w}(k[y_1, \ldots, y_w]/\mathfrak{m}^a)$.

Proof: For brevity, set $G = G_{w_1,\ldots,w_d}$. We can rewrite

$$k[\Sigma_w/G] = k[\Sigma_w]/k[\Sigma_w] \cdot (\{[g] - [g']\}_{g,g' \in G}),$$

so the right hand side of (2.3.1) is the quotient of $\mathrm{CH}_*^{y_1 \cdots y_w}(k[y_1, \ldots, y_w]/\mathfrak{m}^a)$ by the identification $(a_0 \otimes \cdots \otimes a_n)g - (a_0 \otimes \cdots \otimes a_n)g' = 0$ for all $n \geq 0$ and $a_0 \otimes \cdots \otimes a_n \in \mathrm{CH}_n^{y_1 \cdots y_w}(k[y_1, \ldots, y_w]/\mathfrak{m}^a)$. This gives exactly the coinvariants of the G-action, $\mathrm{CH}_*^{y_1 \cdots y_w}(k[y_1, \ldots, y_w]/\mathfrak{m}^a)_G$.

But the left hand side of (2.3.1) is isomorphic to the coinvariants of the G action on $\mathrm{CH}_*^{y_1 \cdots y_w}(k[y_1, \ldots, y_w]/\mathfrak{m}^a)$ too. We will first show using the universal property of coinvariants that for any $n \geq 0$, $\mathrm{CH}_n^{x_1^{w_1} x_2^{w_2} \cdots x_d^{w_d}}(k[x_1, \ldots, x_d]/\mathfrak{m}^a)$ is isomorphic to the coinvariants of the G action on $\mathrm{CH}_n^{y_1 \cdots y_w}(k[y_1, \ldots, y_w]/\mathfrak{m}^a)$. Let N be a trivial G-module, and consider the diagram

$$\begin{array}{ccc} \mathrm{CH}_n^{y_1 \cdots y_w}(k[y_1, \ldots, y_w]/\mathfrak{m}^a) & \xrightarrow{\ f\ } & N \\ \Big\downarrow{\scriptstyle \phi_n} & & \\ \mathrm{CH}_n^{x_1^{w_1} x_2^{w_2} \cdots x_d^{w_d}}(k[x_1, \ldots, x_d]/\mathfrak{m}^a) & & \end{array}$$

where f is any G-equivariant map, and ϕ_n is induced by the ring map $\phi : k[y_1, \ldots, y_w]/\mathfrak{m}^a \to k[x_1, \ldots, x_d]/\mathfrak{m}^a$ sending $y_1, y_2, \ldots, y_{w_1}$ to x_1 and $y_{w_1+1}, y_{w_1+2}, \ldots, y_{w_1+w_2}$ to x_2, etc.

Then there is a unique map $\tilde{f}$: $\mathrm{CH}_n^{x_1^{w_1} x_2^{w_2} \cdots x_d^{w_d}}(k[x_1, \ldots, x_d]/\mathfrak{m}^a) \to N$ such that $f = \tilde{f} \circ \phi_n$. To see this, define a right inverse s of ϕ_n on each tensor of monomials by

$$s(x_{i_{01}} \cdots x_{i_{0k_0}} \otimes x_{i_{11}} \cdots x_{i_{1k_1}} \otimes \cdots \otimes x_{i_{n1}} \cdots x_{i_{nk_n}}) \\ = y_{j_{01}} \cdots y_{j_{0k_0}} \otimes y_{j_{11}} \cdots y_{j_{1k_1}} \otimes \cdots \otimes y_{j_{n1}} \cdots y_{j_{nk_n}}$$

where $\phi(y_{j_{\ell m}}) = x_{i_{\ell m}}$ for every ℓ, m and where

$$\{j_{\ell m}\colon i_{\ell m} = t\} = \\ \{w_1 + w_2 + \cdots + w_{t-1} + 1, w_1 + w_2 + \cdots + w_{t-1} + 2, \ldots, w_1 + w_2 + \cdots + w_t\}$$

for every $1 \leq t \leq d$ to guarantee that we get a tensor monomial in which each y_j occurs exactly once. Then we take this map s which was defined separately on each tensor of monomials and extend by linearity to a right inverse

$$s\colon \mathrm{CH}_n^{x_1^{w_1} x_2^{w_2} \cdots x_d^{w_d}}(k[x_1, \ldots, x_d]/\mathfrak{m}^a) \to \mathrm{CH}_n^{y_1 \cdots y_w}(k[y_1, \ldots, y_w]/\mathfrak{m}^a)$$

of ϕ_n. Note that the maps s we get in different degrees n do *not* fit together to give a chain map.

Set $\tilde{f} = f \circ s$. Then $\tilde{f} \circ \phi_n$ agrees with f on tensors of monomials: we have $\tilde{f} \circ \phi_n = f \circ s \circ \phi_n$, and

$$s \circ \phi_n(y_{j_{01}} \cdots y_{j_{0k_0}} \otimes y_{j_{11}} \cdots y_{j_{1k_1}} \otimes \cdots \otimes y_{j_{n1}} \cdots y_{j_{nk_n}}) \\ = (y_{j_{01}} \cdots y_{j_{0k_0}} \otimes y_{j_{11}} \cdots y_{j_{1k_1}} \otimes \cdots \otimes y_{j_{n1}} \cdots y_{j_{nk_n}})g$$

for some $g \in G$ (which depends on $y_{j_{01}} \cdots y_{j_{0k_0}} \otimes \cdots \otimes y_{j_{n1}} \cdots y_{j_{nk_n}}$) by our definition of s. Since f is equivariant and G acts trivially on N,

$$\begin{aligned} \tilde{f} \circ \phi_n&(y_{j_{01}} \cdots y_{j_{0k_0}} \otimes \cdots \otimes y_{j_{n1}} \cdots y_{j_{nk_n}}) \\ &= f \circ s \circ \phi_n(y_{j_{01}} \cdots y_{j_{0k_0}} \otimes \cdots \otimes y_{j_{n1}} \cdots y_{j_{nk_n}}) \\ &= f((y_{j_{01}} \cdots y_{j_{0k_0}} \otimes \cdots \otimes y_{j_{n1}} \cdots y_{j_{nk_n}})g) \\ &= f(y_{j_{01}} \cdots y_{j_{0k_0}} \otimes \cdots \otimes y_{j_{n1}} \cdots y_{j_{nk_n}})g \\ &= f(y_{j_{01}} \cdots y_{j_{0k_0}} \otimes \cdots \otimes y_{j_{n1}} \cdots y_{j_{nk_n}}). \end{aligned}$$

By linearity we get the desired equality $\tilde{f} \circ \phi_n = f$. The uniqueness of $\tilde{f}$ follows easily from the observation that if we want to have $f = \tilde{f} \circ \phi_n$, we must have $f \circ s = \tilde{f} \circ \phi_n \circ s = \tilde{f} \circ \mathrm{id}_n = \tilde{f}$.

Now, the complex map

$$\phi_*\colon \mathrm{CH}_*^{y_1 \cdots y_w}(k[y_1, \ldots, y_w]/\mathfrak{m}^a) \to \mathrm{CH}_*^{x_1^{w_1} x_2^{w_2} \cdots x_d^{w_d}}(k[x_1, \ldots, x_d]/\mathfrak{m}^a)$$

induces an isomorphism from the coinvariants of the domain onto the range in each dimension n, so the complex $\mathrm{CH}_*^{x_1^{w_1} x_2^{w_2} \cdots x_d^{w_d}}(k[x_1,\ldots,x_d]/\mathfrak{m}^a)$ is isomorphic to the coinvariants of the G action on $\mathrm{CH}_*^{y_1\cdots y_w}(k[y_1,\ldots,y_w]/\mathfrak{m}^a)$. ∎

LEMMA 2.4: *For any $w>0$, there is an isomorphism of left $k[\Sigma_w]$-modules*

$$\bigoplus_{\substack{w_1+\cdots+w_d=w\\ 0\le w_i,\ 1\le i\le d}} k[\Sigma_w/G_{w_1,\ldots,w_d}] \cong (k^d)^{\otimes w}.$$

Proof: Let $e_1,\ldots,e_d$ be the standard basis of k^d. Then

$$(k^d)^{\otimes w} = \bigoplus_{i_1,\ldots,i_w\in\{1,\ldots,d\}} k\cdot(e_{i_1}\otimes e_{i_2}\otimes\cdots\otimes e_{i_w}) =$$

$$\bigoplus_{w_1+\cdots+w_d=w} k[\Sigma_w/G_{w_1,\ldots,w_d}]\cdot(\underbrace{e_1\otimes\cdots\otimes e_1}_{w_1}\otimes\underbrace{e_2\otimes\cdots\otimes e_2}_{w_2}\otimes\cdots\otimes\underbrace{e_d\otimes\cdots\otimes e_d}_{w_d})$$

since the stabilizer of $\underbrace{e_1\otimes\cdots\otimes e_1}_{w_1}\otimes\underbrace{e_2\otimes\cdots\otimes e_2}_{w_2}\otimes\cdots\otimes\underbrace{e_d\otimes\cdots\otimes e_d}_{w_d}$ under the Σ_w action which permutes the tensor factors is exactly $G_{w_1,\ldots,w_d}$. ∎

We are now ready to prove Theorem 2.1. Using Lemma 2.3 for the second isomorphism and Lemma 2.4 for the fourth one, we get

(2.4.1)

$$\mathrm{CH}_*(k[x_1,\ldots,x_d]/\mathfrak{m}^a) \cong \bigoplus_{(w_1,\ldots,w_d)\in\mathbb{N}^d} \mathrm{CH}_*^{x_1^{w_1} x_2^{w_2}\cdots x_d^{w_d}}(k[x_1,\ldots,x_d]/\mathfrak{m}^a)$$

$$\cong \bigoplus_{\substack{(w_1,\ldots,w_d)\in\mathbb{N}^d\\ w=w_1+\cdots+w_d}} \mathrm{CH}_*^{y_1\cdots y_w}(k[y_1,\ldots,y_w]/\mathfrak{m}^a)\otimes_{k[\Sigma_w]} k[\Sigma_w/G_{w_1,\ldots,w_d}]$$

$$\cong \bigoplus_{w=0}^{\infty}\left(\mathrm{CH}_*^{y_1\cdots y_w}(k[y_1,\ldots,y_w]/\mathfrak{m}^a)\otimes_{k[\Sigma_w]} \bigoplus_{w_1+\cdots+w_d=w} k[\Sigma_w/G_{w_1,\ldots,w_d}]\right)$$

$$\cong \bigoplus_{w=0}^{\infty} \mathrm{CH}_*^{y_1\cdots y_w}(k[y_1,\ldots,y_w]/\mathfrak{m}^a)\otimes_{k[\Sigma_w]} (k^d)^{\otimes w}.$$

Since k is a characteristic 0 field, $k[\Sigma_w]$ is a semisimple ring, i.e., all modules over it are projective (see [7]). So tensoring over $k[\Sigma_w]$ with a $k[\Sigma_w]$ module, such as $(k^d)^{\otimes w}$, commutes with taking homology, and (2.1.1) follows from the last line of (2.4.1). ∎

3. The calculation of $\mathrm{HH}_*^{x_1\cdots x_d}(k[x_1,\ldots,x_d]/\mathfrak{m}^a)$

To calculate $\mathrm{HH}_*^{x_1\cdots x_d}(k[x_1,\ldots,x_d]/\mathfrak{m}^a)$, we will set up an explicit correspondence between the reduced Hochschild complex $\widetilde{\mathrm{CH}}_*^{x_1\cdots x_d}(k[x_1,\ldots,x_d])$ and the cellular chain complex of a particular cell decomposition of the d-torus $\mathbb{T}^d$. The cells will, in fact, have the linear structure of simplices and we will be able to define a simplicial-type boundary operation $\partial=\sum_{i=0}^n(-1)^i d_i$ which will agree with the ∂ (and in fact even with the d_i) of $\widetilde{\mathrm{CH}}_*^{x_1\cdots x_d}(k[x_1,\ldots,x_d])$. This cell decomposition is not a triangulation only because the combinatorial requirement of each pair of simplices intersecting along a simplex or not at all does not hold (there is, for example, only one vertex).

The cellular decomposition of $\mathbb{T}^d=(\mathbb{R}/\mathbb{Z})^d$ is the d-fold product of the standard simplicial structure of the circle S^1. Explicitly: for any ordered list of sets $P_0,P_1,\ldots,P_n$ where P_0 is possibly empty but the others are not, and where $\bigcup_{i=0}^n P_i=\{1,2,\ldots,d\}$ and $P_i\cap P_j=\emptyset$ whenever $i\neq j$, we look at

$$\sigma_{P_0,P_1,\ldots,P_n}=\left\{\begin{array}{c}(t_1,\ldots,t_d)\in\mathbb{T}^d\text{ s.t. there exist }0=\alpha_0\le\alpha_1\le\cdots\le\alpha_n\le 1\\ \text{so that for all }0\le i\le n\text{ and all }\ell\in P_i,\ t_\ell=\alpha_i\end{array}\right\}.$$

This $\sigma_{P_0,P_1,\ldots,P_n}$ is isomorphic to a closed n-simplex with some identifications on its boundary, which result from the identification of 0 and 1 in $\mathbb{R}/\mathbb{Z}$. Its $(n-1)$-dimensional faces are

$$\sigma_{P_0\cup P_1,P_2,\ldots,P_n},\sigma_{P_0,P_1\cup P_2,\ldots,P_n},\ldots,\sigma_{P_0,P_1,\ldots,P_{n-1}\cup P_n},\sigma_{P_n\cup P_0,P_1,\ldots,P_{n-1}}.$$

These $(n-1)$-faces correspond to elements of $\sigma_{P_0,P_1,\ldots,P_n}$ in which two consecutive α_i become equal (in the case of the last face listed, to elements in which $\alpha_n=1=0=\alpha_0$).

Map the standard closed n-simplex onto $\sigma_{P_0,P_1,\ldots,P_n}$ so the ith face will map to $\sigma_{P_0,P_1,\ldots,P_i\cup P_{i+1},\ldots,P_n}$ for each $0\le i<n$ and the nth face will map to $\sigma_{P_n\cup P_0,P_1,\ldots,P_{n-1}}$. Then if we look at the cellular chain complex for this cellular decomposition,

$$\partial\sigma_{P_0,P_1,\ldots,P_n}=\sum_{i=0}^{n-1}(-1)^i\sigma_{P_0,\ldots,P_i\cup P_{i+1},\ldots,P_n}+(-1)^n\sigma_{P_n\cup P_0,P_1,\ldots,P_{n-1}}. \tag{3.0.1}$$

For any ordered list of sets $P_0,P_1,\ldots,P_n$ where P_0 is possibly empty but the others are not, and where $\bigcup_{i=0}^n P_i=\{1,2,\ldots,d\}$ and $P_i\cap P_j=\emptyset$ whenever $i\neq j$, we can also look at the tensor monomial

$$m_{P_0,P_1,\ldots,P_n}=\prod_{\ell\in P_0}x_\ell\otimes\prod_{\ell\in P_1}x_\ell\otimes\cdots\otimes\prod_{\ell\in P_n}x_\ell\in\widetilde{\mathrm{CH}}_n^{x_1\cdots x_d}(k[x_1,\ldots,x_d]). \tag{3.0.2}$$

The $m_{P_0,P_1,\ldots,P_n}$ form a basis for $\widetilde{\mathrm{CH}}_n^{x_1\cdots x_d}(k[x_1,\ldots,x_d])$ over k, just as the $\sigma_{P_0,P_1,\ldots,P_n}$ form a basis for the cellular n-chains on $\mathbb{T}^d$, $C_n(\mathbb{T}^d;k)$. The Hochschild homology boundary corresponds exactly to the cellular boundary (3.0.1), so the complexes are isomorphic.

This discussion can be regarded as a fancy proof for the fact that

$$\mathrm{HH}_*^{x_1\cdots x_d}(k[x_1,\ldots,x_d]) \cong \mathrm{H}_*(\mathbb{T}^d;k),$$

a fact which follows more easily from the observation that each side is isomorphic to the dth tensor power of $\mathrm{HH}_*^x(k[x]) \cong \mathrm{H}_*(S^1;k)$.

It also allows us to see at the level of $\mathbb{T}^d$ what happens to the Hochschild homology when we take the quotient of $k[x_1,\ldots,x_d]$ by $\mathfrak{m}^a$. In the $x_1\cdots x_d$ summand of the reduced Hochschild complex, quotienting by $\mathfrak{m}^a$ means setting $m_{P_0,P_1,\ldots,P_n} \equiv 0$ whenever $|P_i| \geq a$ for some $0 \leq i \leq n$. If we did the analogous thing on our cellular chains $C_n(\mathbb{T}^d;k)$, we would identify with 0 every $\sigma_{P_0,P_1,\ldots,P_n}$ where $|P_i| \geq a$ for some $0 \leq i \leq n$. Define

$$\Delta_a = \{(t_1,\ldots,t_d) \in \mathbb{T}^d \text{ such that } |\{i\colon t_i = \alpha\}| \geq a \text{ for some } \alpha\}.$$

If $a = 2$, Δ_a is the fat diagonal, consisting of all points on the torus whose coordinates are not all different from each other; if $a = d$, it is the usual diagonal; if $a > d$, $\Delta_a = \emptyset$. The subspace Δ_a is a subcomplex of our cellular decomposition of $\mathbb{T}^d$.

This shows that $\widetilde{\mathrm{CH}}_*^{x_1\cdots x_d}(k[x_1,\ldots,x_d]/\mathfrak{m}^a)$ is isomorphic to the relative cellular chain complex $C_*(\mathbb{T}^d,\Delta_a;k)$, and we have proved

Theorem 3.1: *If k is a field of characteristic 0 and $\mathfrak{m} = (x_1,\ldots,x_d)$,*

$$\mathrm{HH}_*^{x_1\cdots x_d}(k[x_1,\ldots,x_d]/\mathfrak{m}^a) \cong \mathrm{H}_*(\mathbb{T}^d,\Delta_a;k). \qquad \blacksquare$$

Corollary 3.2: *If k is a field of characteristic 0 and $\mathfrak{m} = (x_1,\ldots,x_d)$, there is an algebra isomorphism*

$$\mathrm{HH}_*(k[x_1,\ldots,x_d]/\mathfrak{m}^a) \cong \bigoplus_{w=0}^{\infty} \mathrm{H}_*(\mathbb{T}^w,\Delta_a;k) \otimes_{k[\Sigma_w]} (k^d)^{\otimes w}$$

where the product on Hochschild homology is induced by the shuffle product, and the product on the right hand side is induced by the map of pairs $(\mathbb{T}^w,\Delta_a) \times (\mathbb{T}^{w'},\Delta_a) \to (\mathbb{T}^{w+w'},\Delta_a)$ coming from the standard identification $\mathbb{T}^w \times \mathbb{T}^{w'} \cong \mathbb{T}^{w+w'}$, tensored with the identification $(k^d)^{\otimes w} \otimes (k^d)^{\otimes w'} \cong (k^d)^{\otimes w+w'}$.

Proof: The existence of a linear isomorphism follows by putting together Theorem 2.1 and Theorem 3.1; we need to show that it is in fact an algebra isomorphism. The shuffle product

$$\begin{aligned}\mathrm{CH}_*^{x_1^{w_1}x_2^{w_2}\cdots x_d^{w_d}}(k[x_1,\ldots,x_d]/\mathfrak{m}^a)\otimes\mathrm{CH}_*^{x_1^{w'_1}x_2^{w'_2}\cdots x_d^{w'_d}}(k[x_1,\ldots,x_d]/\mathfrak{m}^a)\\ \to\mathrm{CH}_*^{x_1^{w_1+w'_1}x_2^{w_2+w'_2}\cdots x_d^{w_d+w'_d}}(k[x_1,\ldots,x_d]/\mathfrak{m}^a)\end{aligned}$$

is given, in terms of the isomorphism of Lemma 2.3, as the shuffle product

$$\begin{aligned}&\mathrm{CH}_*^{y_1\cdots y_w}(k[y_1,\ldots,y_w]/\mathfrak{m}^a)_{G_{w_1,\ldots,w_d}}\\ (3.2.1)\qquad&\otimes\mathrm{CH}_*^{y_{w+1}\cdots y_{w+w'}}(k[y_{w+1},\ldots,y_{w+w'}]/\mathfrak{m}^a)_{G_{w'_1,\ldots,w'_d}}\\ &\to\mathrm{CH}_*^{y_1\cdots y_{w+w'}}(k[y_1,\ldots,y_{w+w'}]/\mathfrak{m}^a)_{G_{w_1,\ldots,w_d}\times G_{w'_1,\ldots,w'_d}}\end{aligned}$$

followed by the quotient map

$$\begin{aligned}(3.2.2)\qquad&\mathrm{CH}_*^{y_1\cdots y_{w+w'}}(k[y_1,\ldots,y_{w+w'}]/\mathfrak{m}^a)_{G_{w_1,\ldots,w_d}\times G_{w'_1,\ldots,w'_d}}\\ &\to\mathrm{CH}_*^{y_1\cdots y_{w+w'}}(k[y_1,\ldots,y_{w+w'}]/\mathfrak{m}^a)_{G_{w_1+w'_1,\ldots,w_d+w'_d}}\end{aligned}$$

where $w=w_1+w_2+\cdots+w_d$ as usual and $w'=w'_1+w'_2+\cdots+w'_d$.

By the isomorphism described below (3.0.2), the shuffle product (3.2.1) corresponds to the prismic subdivision Alexander–Whitney map

$$C_*(\mathbb{T}^w,\Delta_a;k)\otimes C_*(\mathbb{T}^{w'},\Delta_a;k)\to C_*(\mathbb{T}^{w+w'},\Delta_a;k)$$

coming from our chosen simplicial structures and the standard identification $\mathbb{T}^w\times\mathbb{T}^{w'}\cong\mathbb{T}^{w+w'}$, tensored with the product

$$k[\Sigma_w/G_{w_1,\ldots,w_d}]\otimes k[\Sigma_{w'}/G_{w'_1,\ldots,w'_d}]\to k[\Sigma_{w+w'}/G_{w_1,\ldots,w_d}\times G_{w'_1,\ldots,w'_d}]$$

coming from the identification

$$(k^d)^{\otimes w}\otimes(k^d)^{\otimes w'}\cong(k^d)^{\otimes w+w'}$$

and the inclusions of Lemma 2.4. ∎

4. Recovering the Hochschild homology of dual numbers

When $d=1$, $k[x]/x^a$ is a complete intersection, and its Hochschild homology is well-known. For example, in the simplest of these cases, the dual numbers

$k[x]/x^2$, the Hochschild homology is known to be one-dimensional in every positive degree and two-dimensional in degree zero. It is easy to recover this result from Corollary 3.2:

For $w \geq 1$, $\mathbb{T}^w$ consists of the fat diagonal Δ_2, the $(w-1)$-cells

$$\sigma_\gamma = \sigma_{\{\gamma(1)\},\{\gamma(2)\},\ldots,\{\gamma(w)\}}, \quad \gamma \in \Sigma_w,$$

and the w-cells

$$\sigma_\gamma^0 = \sigma_{\emptyset,\{\gamma(1)\},\{\gamma(2)\},\ldots,\{\gamma(w)\}}, \quad \gamma \in \Sigma_w.$$

The boundary ∂ of the complex $C_*(\mathbb{T}^w, \Delta_2; k)$ satisfies

$$\partial(\sigma_\gamma) = 0, \quad \partial(\sigma_\gamma^0) = \sigma_\gamma + (-1)^w \sigma_{\gamma\circ\tau}$$

for all $\gamma \in \Sigma_w$, where τ is the w-cycle $(w\ w-1\ \cdots 2\ 1)$. So the complex $C_*(\mathbb{T}^w, \Delta_2; k)$ is isomorphic over $k[\Sigma_w]$ to the complex

$$\cdots \to 0 \to 0 \to k[\Sigma_w] \overset{\mathrm{id}+(-1)^w\tau}{\longrightarrow} k[\Sigma_w] \to 0 \to 0 \to \cdots.$$

Since k is a field of characteristic 0, it is a projective $k[\Sigma_w]$-module, so tensoring with it commutes with taking homology. Thus $H_*(\mathbb{T}^w, \Delta_2) \otimes_{k[\Sigma_w]} k$ is the homology of the complex

$$\cdots \to 0 \to 0 \to k \overset{\mathrm{id}+(-1)^w\mathrm{id}}{\longrightarrow} k \to 0 \to 0 \to \cdots,$$

which is trivial when w is even but gives one copy of k in degree $w-1$ and another in degree w when w is odd.

For $w = 0$, $H_*(pt, \emptyset; k) \otimes_k k$ contributes an additional k in degree zero.

The weight decomposition allows us to see the multiplicative structure very easily in this case (a general, explicit calculation of shuffle products of representatives of homology classes in extensions of the form $k[x]/f(x)$ over k is given in Proposition 1.15 of [5]). The weight zero element $1 \cdot pt \otimes 1$ is the unit for the multiplication. Products of elements in the higher weight summands all vanish: all the nontrivial homology classes have odd weights, so their products must have even weight and thus be trivial.

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